

The Non-Abelian Topological Gauge Field Theory of \tilde{p} -Branes ¹

Yi-Shi Duan ^a, Ji-Rong Ren ^{a,*}

^a*Institute of Theoretical Physics,
School of Physical Science and Technology,
Lanzhou University, Lanzhou, 730000, P. R. China*

Abstract

By the generalization of Chern–Simons topological current and Gauss–Bonnet–Chern theorem, the purpose of this paper is to make a non-Abelian gauge field theory foundation of the topological current of \tilde{p} -branes formulated in our previous work. Using ϕ -mapping topological current theory proposed by Professor Duan, we find that the topological \tilde{p} -branes are created at every isolated zero of vector field $\vec{\phi}(x)$. It is shown that the topological charges carried by \tilde{p} -branes are topologically quantized and labeled by Hopf index and Brouwer degree, i.e., the winding number of the ϕ -mapping. The action of topological \tilde{p} -branes is obtained and is just Nambu action for multistrings when $D - \tilde{d} = 2$.

Key words: \tilde{p} -branes, Non-Abelian Topological Gauge Field Theory, Gauss-Bonnet-Chern theorem, ϕ -mapping topological current theory
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1 Introduction

Extended objects with p spatial dimension, known as p -branes, play an essential role in revealing the nonperturbative structure of the superstring theories and M-theories[1–6]. Antisymmetric tensor gauge fields determine all of the features of a p -brane and have been widely studied in the theory of p -branes

* Corresponding author

Email address: renjr@lzu.edu.cn (Ji-Rong Ren).

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[7–10,12]. In the context of the effective $D = 10$ or $D = 11$ supergravity theory a p -brane is a p -dimensional extended source for a $(p + 2)$ -form gauge field strength F . It is well-known that the $(p + 2)$ -form strength F satisfies the field equation

$$\nabla_\mu F^{\mu\mu_1\cdots\mu_{p+1}} = j^{\mu_1\cdots\mu_{p+1}} \quad (1)$$

where $j^{\mu_1\cdots\mu_{p+1}}$ is a $(p + 1)$ -form tensor current and corresponding to "electric source", and the dual field strength $*F$ satisfies

$$\nabla_\mu *F^{\mu\mu_1\cdots\mu_{\tilde{p}+1}} = \tilde{j}^{\mu_1\cdots\mu_{\tilde{p}+1}} \quad (2)$$

in which $\tilde{j}^{\mu_1\cdots\mu_{\tilde{p}+1}}$ is a extended $(\tilde{p} + 1)$ -form topological tensor current and corresponding to "magnetic source" [11–13]. In Ref.[7,11,12,14,15], from the perspective of a higher dimensional theory, the topological theories of \tilde{p} -branes in M -theory were also studied. The ϕ -mapping topological current theory proposed by Professor Duan plays a crucial role in studying the structure of topological defects[16–30]. In our previous work[28], using the ϕ -mapping topological current theory, we had present a new topological tensor current of \tilde{p} -branes. It's shown that the current is identically conserved and behaves as $\delta(\vec{\phi})$, and every isolated zero of the field $\vec{\phi}(x)$ corresponding to a "magnetic" \tilde{p} -brane. It must be pointed out that usually the study of extended \tilde{p} -branes is always via generalizing Kalb-Ramond Abelian gauge field[7,33]. That is a kind of $U(1)$ gauge field theory, so far from which the topological current can't be strictly induced. In fact the present work is a generalization of GBC topological current of moving point defect[19,20]. By making use of the generalization of Gauss-Bonnet-Chern theorem and ϕ -mapping field theory, we find a $SO(N)$ non-Abelian topological field theory of \tilde{p} -branes, in which the dual field strength $*F^{\mu\mu_1\cdots\mu_{\tilde{p}+1}}$ of eq.(2) can rigorously create a topological current of \tilde{p} -branes in a natural way. In the $SO(N)$ non-Abelian topological gauge field theory of \tilde{p} -branes, we also investigate the inner structure of topological current of \tilde{p} -branes, show that the topological charges of \tilde{p} -branes are topologically quantized and labeled by the Hopf index and Brouwer degree, the winding number of the ϕ -mapping. We also find that in this $SO(N)$ gauge field theory of \tilde{p} -branes, when N is even, the topological tensor current of \tilde{p} -branes $\tilde{j}^{\mu_1\cdots\mu_{\tilde{p}+1}}$ can be looked upon as the generalization of Chern-Simoin topological current, that we have formulated in [28].

2 The non-Abelian field theory of \tilde{p} -branes

In this paper the studies of the non-Abelian gauge field theory and topological current of \tilde{p} -branes are basing on the Gauss-Bonnet-Chern(GBC) theorem

and the generalization of Chern–Simons topological current. It is well-known that Gauss–Bonnet–Chern theorem is a generalization of Euler number density from two dimensional Gauss–Bonnet theorem to arbitrary even dimensional theory, which relates the curvature of the compact and oriented even-dimensional Riemannian manifold M with an important topological invariant, the Euler–Poincaré characteristic $\chi(M)$. The *GBC*–form corresponding to Euler number density is given by

$$\Lambda = \frac{(-1)^{\frac{N}{2}-1}}{2^N \pi^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \varepsilon_{A_1 A_2 \dots A_{N-1} A_N} F^{A_1 A_2} \wedge \dots \wedge F^{A_{N-1} A_N}, \quad (3)$$

in which F^{AB} is the curvature tensor of $SO(N)$ principal bundle of the Riemannian manifold M , i.e., the $SO(N)$ gauge field 2–form

$$F^{AB} = d\omega^{AB} - \omega^{AC} \wedge \omega^{CB}. \quad (4)$$

where ω^{AB} is the spin connection 1-form. In 1944, an elegantly intrinsic proof of the theorem was given by Chern[34], whose instructive idea was to work on the sphere bundle $S^{N-1}(M)$. Using a recursion method Chern has proved that the *GBC*–form is exact on $S^{N-1}(M)$

$$\Lambda = d\Omega, \quad (5)$$

where the $(N-1)$ –form Ω is called the *Chern*–form

$$\Omega = \frac{1}{(2\pi)^{\frac{N}{2}}} \sum_{k=0}^{\frac{N}{2}-1} (-1)^k \frac{2^{-k}}{(N-2k-1)!!k!} \Theta_k, \quad (6)$$

in which

$$\begin{aligned} \Theta_k = & \varepsilon_{A_1 A_2 \dots A_{N-2k} A_{N-2k+1} A_{N-2k+2} \dots A_{N-1} A_N} n^{A_1} \theta^{A_2} \wedge \dots \\ & \wedge \theta^{A_{N-2k}} \wedge F^{A_{N-2k+1} A_{N-2k+2}} \wedge \dots \wedge F^{A_{N-1} A_N}, \end{aligned} \quad (7)$$

$$\theta^A \equiv Dn^A = dn^A - \omega^{AB} n^B, \quad (8)$$

and n^A is the section of the sphere bundle $S^{N-1}(M)$

$$n : \partial M \rightarrow S^{N-1}(M). \quad (9)$$

A detailed review of Chern's proof of the *GBC* theorem was presented in Ref.[35] and one great advance in this field is the discovery of the relationship between supersymmetry and the index theorem[36].

It must be pointed out that the *GBC* theorem is formulated by the exterior differential forms[34]. Differential forms are a vector space (with a C-infinity topology) and therefore have a dual space in higher-dimension space. Submanifolds represent an element of the dual via integration, so it is common to say that they are in the dual space of forms, which is the space of currents[37]. let (X, g) be a D -dimensional manifold and $F_{\mu\nu}^{AB}$ the curvature tensor of $SO(N)$ principal bundle, we can define a $(\tilde{p}+1)$ -dimensional topological tensor current on manifold X

$$\tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} = \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\mu_2\cdots\mu_N}}{\sqrt{g}} \frac{(-1)^{\frac{N}{2}-1} N!}{2^N (2\pi)^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \cdot \varepsilon_{A_1 A_2 \cdots A_{N-1} A_N} F_{\mu_1 \mu_2}^{A_1 A_2} \cdots F_{\mu_{N-1} \mu_N}^{A_{N-1} A_N}, \quad (10)$$

where N is the dimension of a submanifold M . It is easy to see that eq.(10) is just the generalization of Chern-Simons $SO(2)$ topological current[27]

$$\tilde{j}^\lambda = \frac{1}{8\pi} \frac{\varepsilon^{\lambda\mu\nu}}{\sqrt{g}} \varepsilon_{AB} F_{\mu\nu}^{AB}. \quad (11)$$

Using the Bianchi identity,

$$D_\mu F_{\nu\lambda}^{AB} + D_\nu F_{\lambda\mu}^{AB} + D_\lambda F_{\mu\nu}^{AB} = 0 \quad (12)$$

one find

$$\varepsilon^{\mu_1 \cdots \mu_{i-1} \mu_i \mu_{i+1} \cdots \mu_{N+k}} D_{\mu_{i-1}} F_{\mu_i \mu_{i+1}}^{AB} = 0. \quad (13)$$

From (10), it can be proved[38] that

$$\nabla_\lambda \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} = 0, \quad (14)$$

i.e., the antisymmetric topological tensor current $\tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}}$ is identically conserved.

It's also easy to find that the topological tensor current $\tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}}$ is the dual tensor of *GBC* tensor defined in eq.(3)

$$\tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} = \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\mu_2\cdots\mu_N}}{\sqrt{g}} \Lambda_{\mu_1\mu_2\cdots\mu_N}. \quad (15)$$

Furthermore, by virtue of the tensor form of *GBC* theorem (3) and (5)

$$\Lambda_{\mu_1\mu_2\cdots\mu_N} = \partial_{[\mu_1} F_{\mu_2\cdots\mu_N]}, \quad (16)$$

we can find that the dual tensor of *Chern*-tensor $F_{\mu_2\cdots\mu_N}$ is

$$*F^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1} = \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\mu_2\cdots\mu_N}}{\sqrt{g}} F_{\mu_2\cdots\mu_N}. \quad (17)$$

It's obvious that if in eq.(2) the dual field tensor $*F^{\mu\mu_1\cdots\mu_{\tilde{p}+1}}$ is taking the form of (17) deduced from *GBC* theorem, then we have the conserved topological tensor current (10). Therefore for the case of even N the anti-symmetric tensor current(10) that constructed in term of $SO(N)$ gauge field tensor $F_{\mu\nu}^{AB}$ is just the topological current of creating \tilde{p} -branes. As to the field tensor $F^{\mu\mu_1\cdots\mu_{\tilde{p}+1}}$ in (1) and the dual tensor $*F^{\mu\mu_1\cdots\mu_{\tilde{p}+1}}$ in (2) can be found by making use of the Chern-form. In the following we will show that the tensor current defined by (10) is just the ϕ -mapping topological tensor current of \tilde{p} -branes in [28]. This is a novel foundation of the non-Abelian topological gauge field theory of \tilde{p} -branes.

The early work of Chern[34] had shown that in a neighborhood of arbitrary point P on M it can be chosen a family of frames such that the spin connection $\omega^{AB} = 0$. This locally Euclidean homeomorphism immediately gives an important consequence of the *GBC* theorem on sphere bundle $S^{N-1}(X)$:

$$\Omega = \frac{1}{(2\pi)^{\frac{N}{2}}} \frac{1}{(N-1)!!} \varepsilon_{A_1 A_2 \cdots A_N} n^{A_1} dn^{A_2} \wedge \cdots \wedge dn^{A_N}. \quad (18)$$

Using the unit sphere area formula $A(S^{N-1}) = 2\pi^{N/2}/\Gamma(\frac{N}{2})$ and the following relation

$$(2\pi)^{\frac{N}{2}} (N-1)!! = A(S^{N-1})(N-1)!, \quad (19)$$

the *Chern*-form expressed by (18) can be locally reduced to

$$\Omega = \frac{1}{A(S^{N-1})(N-1)!} \varepsilon_{A_1 A_2 \cdots A_N} n^{A_1} dn^{A_2} \wedge \cdots \wedge dn^{A_N}. \quad (20)$$

We see that the expression (20) is nothing but the ratio of area element to the total area $A(S^{N-1})$ of unit sphere S^{N-1} . This is essential of *GBC* theorem.

Using the *Chern*-form(20), it's easy to prove that the *Chern* tensor field can be simply written as

$$F_{\mu_2 \dots \mu_N} = \frac{1}{A(S^{N-1})(N-1)!} \varepsilon_{A_1 A_2 \dots A_N} n^{A_1} \partial_{\mu_2} n^{A_2} \dots \partial_{\mu_N} n^{A_N} \quad (21)$$

and the $(\tilde{p}+1)$ -dimensional tensor current (10) can also be expressed as follows

$$\begin{aligned} \tilde{j}^{\lambda \lambda_1 \dots \lambda_{\tilde{p}}} &= \frac{1}{A(S^{N-1})(N-1)!} \varepsilon_{A_1 A_2 \dots A_{N-1} A_N} \\ &\cdot \frac{\varepsilon^{\lambda \lambda_1 \dots \lambda_{\tilde{p}} \mu_1 \mu_2 \dots \mu_N}}{\sqrt{g}} \partial_{\mu_1} n^{A_1} \dots \partial_{\mu_N} n^{A_N}. \end{aligned} \quad (22)$$

This is just the topological tensor current of \tilde{p} -branes in [28], i.e., the tensor current of creating \tilde{p} -dimensional manifold[31,32].

In case of $SO(N+1)$ gauge field theory on D -dimensional manifold X , we can define a new field theory as

$$\begin{aligned} F_{\mu_1 \dots \mu_N} &= \frac{(-1)^{\frac{N}{2}-1} N!}{2^N (2\pi)^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \varepsilon_{A A_1 A_2 \dots A_{N-1} A_N} \\ &\cdot n^A F_{\mu_1 \mu_2}^{A_1 A_2} \dots F_{\mu_{N-1} \mu_N}^{A_{N-1} A_N}, \end{aligned} \quad (23)$$

$$\star F^{\lambda \lambda_1 \dots \lambda_{\tilde{p}} \mu} = \frac{\varepsilon^{\lambda \lambda_1 \dots \lambda_{\tilde{p}} \mu \mu_1 \mu_2 \dots \mu_N}}{\sqrt{g}} F_{\mu_1 \mu_2 \dots \mu_N}, \quad (24)$$

where n^A is the section of sphere bundle $S^N(X)$. Using (23) and the Bianchi identity (13), we find that the topological tensor current can be defined as

$$\begin{aligned}
& \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\bar{p}}} \\
&= \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} {}^*F^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu} \right) \\
&= \frac{1}{\sqrt{g}} \partial_\mu \left(\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\mu_2\cdots\mu_N} F_{\mu_1\mu_2\cdots\mu_N} \right) \\
&= \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\mu_2\cdots\mu_N}}{\sqrt{g}} D_\mu (F_{\mu_1\mu_2\cdots\mu_N}) \\
&= \frac{(-1)^{\frac{N}{2}-1} N!}{2^N (2\pi)^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \varepsilon_{AA_1A_2\cdots A_{N-1}A_N} \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\cdots\mu_N}}{\sqrt{g}} \\
&\quad \cdot D_\mu n^A F_{\mu_1\mu_2}^{A_1A_2} \cdots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N}.
\end{aligned} \tag{25}$$

In the like manner, it can be proved that the covariant divergence of this topological tensor current is

$$\begin{aligned}
\nabla_\lambda \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\bar{p}}} &= \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\cdots\mu_N}}{\sqrt{g}} \\
&\quad \cdot D_\lambda D_\mu n^A F_{\mu_1\mu_2}^{A_1A_2} \cdots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N}.
\end{aligned} \tag{26}$$

Using the definition of curvature tensor $F_{\mu\nu}^{AB}$ of $SO(N+1)$ principal bundle

$$(D_\mu D_\nu - D_\nu D_\mu) n^A = -F_{\mu\nu}^{AB} n^B, \tag{27}$$

we can obtain

$$\begin{aligned}
\nabla_\lambda \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\bar{p}}} &= \frac{(-1)^{\frac{N}{2}} N!}{2^{(N+1)} (2\pi)^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \varepsilon_{AA_1A_2\cdots A_{N-1}A_N} \\
&\quad \cdot \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\cdots\mu_N}}{\sqrt{g}} F_{\lambda\mu}^{AB} n^B F_{\mu_1\mu_2}^{A_1A_2} \cdots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N},
\end{aligned} \tag{28}$$

where $B, A, A_1, A_2, \cdots, A_{N-1}, A_N$ are valued to $SO(N+1)$ Lie algebra index. Let

$$\Delta^B = \varepsilon_{AA_1A_2\cdots A_{N-1}A_N} \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\bar{p}}\mu\mu_1\cdots\mu_N}}{\sqrt{g}}$$

$$\cdot F_{\lambda\mu}^{AB} F_{\mu_1\mu_2}^{A_1A_2} \dots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N}, \quad (29)$$

for fixed B , if $A = B$, it's obvious that $\Delta^B = 0$; then one of $A_1, A_2, \dots, A_{N-1}, A_N$ must be valued to B and

$$\begin{aligned} \Delta^B &= \varepsilon_{AA_1A_2\dots A_{N-1}A_N} \frac{\varepsilon^{\lambda\lambda_1\dots\lambda_{\tilde{p}}\mu\mu_1\dots\mu_N}}{\sqrt{g}} \\ &\quad \cdot F_{\lambda\mu}^{AB} \dots F_{\mu_i\mu_{i+1}}^{A_iB} \dots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N}, \end{aligned} \quad (30)$$

or

$$\begin{aligned} \Delta^B &= \varepsilon_{AA_1A_2\dots A_{N-1}A_N} \frac{\varepsilon^{\lambda\lambda_1\dots\lambda_{\tilde{p}}\mu\mu_1\dots\mu_N}}{\sqrt{g}} \\ &\quad \cdot F_{\lambda\mu}^{AB} \dots F_{\mu_i\mu_{i+1}}^{BA_{i+1}} \dots F_{\mu_{N-1}\mu_N}^{A_{N-1}A_N}, \end{aligned} \quad (31)$$

where $F_{\lambda\mu}^{AB}$ and $F_{\mu_i\mu_{i+1}}^{A_iB}$, or $F_{\lambda\mu}^{AB}$ and $F_{\mu_i\mu_{i+1}}^{BA_{i+1}}$ can be exchanged symmetrically, but the exchange between A and A_i or A and A_{i+1} is antisymmetric, so we obtain that Δ^B equals to zero also. Therefore, the $SO(N+1)$ topological tensor current $\tilde{j}^{\lambda\lambda_1\dots\lambda_{\tilde{p}}}$ is identically conserved

$$\nabla_\lambda \tilde{j}^{\lambda\lambda_1\dots\lambda_{\tilde{p}}} = 0. \quad (32)$$

In the case of odd N the dual field tensor $*F^{\mu\mu_1\dots\mu_{\tilde{p}+1}}$ in (2) can be directly expressed in term of the field tensor $F_{\mu\nu}^{AB}$ as expression (24).

As above, the eq.(25) can be locally written in a simple form

$$\begin{aligned} \tilde{j}^{\lambda\lambda_1\dots\lambda_{\tilde{p}}} &= \frac{1}{2A(S^{N-1})(N-1)!} \varepsilon_{AA_1A_2\dots A_N} \\ &\quad \cdot \frac{\varepsilon^{\lambda\lambda_1\dots\lambda_{\tilde{p}}\mu\mu_1\dots\mu_N}}{\sqrt{g}} \partial_\mu n^A \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \dots \partial_{\mu_N} n^{A_N}. \end{aligned} \quad (33)$$

Multiplying (25) and (33) by $2A(S^{N-1})/NA(S^N)$ and defining

$$d = \begin{cases} N, & \text{for } d \text{ even} \\ N+1, & \text{for } d \text{ odd} \end{cases} \quad (34)$$

where N is even and $D = d + \tilde{p} + 1$ is the dimension of total manifold X , then we can obtain a unified topological tensor current on D -dimensional smooth manifold X

$$\begin{aligned} \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} &= \frac{1}{A(S^{d-1})(d-1)!} \varepsilon_{A_1 A_2 \cdots A_{d-1} A_d} \\ &\cdot \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\mu_2\cdots\mu_d}}{\sqrt{g}} \partial_{\mu_1} n^{A_1} \cdots \partial_{\mu_d} n^{A_d}. \end{aligned} \quad (35)$$

Here it's easier to prove that above topological currents are locally conserved

$$\nabla_\lambda \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} = \frac{1}{\sqrt{g}} \partial_\lambda \left(\sqrt{g} \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} \right) = 0. \quad (36)$$

It's well known that the ϕ -mapping is a d -dimensional smooth vector field on X

$$\phi^A = \phi^A(x), \quad A = 1, 2, \dots, d, \quad (37)$$

and the direction field of $\vec{\phi}(x)$ is

$$n^A = \frac{\phi^A}{\|\phi\|}, \quad \|\phi\| = \sqrt{\phi^A \phi^A}, \quad (38)$$

i.e., n^A is the section of sphere bundle $S^{d-1}(X)$. Substituting (38) into (35) and considering that

$$\partial_\mu n^A = \frac{\partial_\mu \phi^A}{\|\phi\|} + \phi^A \partial_\mu \left(\frac{1}{\|\phi\|} \right), \quad (39)$$

we have

$$\begin{aligned} \tilde{j}^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}}(x) &= \frac{1}{A(S^{d-1})(d-1)!} \varepsilon_{A_1 \cdots A_d} \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\cdots\mu_d}}{\sqrt{g}} \\ &\cdot \partial_{\mu_1} \left(\frac{\phi^{A_1}}{\|\phi\|^d} \partial_{\mu_2} \phi^{A_2} \cdots \partial_{\mu_d} \phi^{A_d} \right) \\ &= \frac{1}{A(S^{d-1})(d-1)!} \varepsilon_{A_1 \cdots A_d} \frac{\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\cdots\mu_d}}{\sqrt{g}} \\ &\cdot \partial_{\mu_1} \left(\frac{\phi^{A_1}}{\|\phi\|^d} \right) \partial_{\mu_2} \phi^{A_2} \cdots \partial_{\mu_d} \phi^{A_d}. \end{aligned} \quad (40)$$

Defining the $rank-(\tilde{p}+1)$ Jacobian tensor $J^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} \left(\frac{\phi}{x} \right)$ of $\vec{\phi}$ as

$$\begin{aligned} &\varepsilon^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}\mu_1\cdots\mu_d} \partial_{\mu_1} \phi^A \partial_{\mu_2} \phi^{A_2} \cdots \partial_{\mu_d} \phi^{A_d} \\ &= J^{\lambda\lambda_1\cdots\lambda_{\tilde{p}}} \left(\frac{\phi}{x} \right) \varepsilon^{AA_2\cdots A_d}, \end{aligned} \quad (41)$$

and noticing

$$\varepsilon_{A_1 A_2 \dots A_d} \varepsilon^{A A_2 \dots A_d} = \delta_{A_1}^A (d-1)!, \quad (42)$$

it follows that

$$\tilde{j}^{\lambda\lambda_1 \dots \lambda_{\tilde{p}}}(x) = \frac{1}{A(S^{d-1})\sqrt{g}} \frac{\partial}{\partial \phi^A} \left(\frac{\phi^A}{\|\phi\|^d} \right) J^{\lambda\lambda_1 \dots \lambda_{\tilde{p}}} \left(\frac{\phi}{x} \right). \quad (43)$$

Using the Green functions in $\vec{\phi}$ -space

$$\frac{\phi^A}{\|\phi\|^d} = \begin{cases} -\frac{1}{(d-2)} \frac{\partial}{\partial \phi^A} \left(\frac{1}{\|\phi\|^{d-2}} \right) & \text{for } d > 2, \\ \frac{\partial}{\partial \phi^A} \ln \|\phi\| & \text{for } d = 2, \end{cases} \quad (44)$$

and

$$\Delta_\phi \left(\frac{1}{\|\phi\|^{d-2}} \right) = -(d-2) A(S^{d-1}) \delta(\vec{\phi}), \quad (45)$$

$$\Delta_\phi (\ln \|\phi\|) = 2\pi \delta(\vec{\phi}), \quad (46)$$

where $\Delta_\phi = \frac{\partial^2}{\partial \phi^A \partial \phi^A}$ is the d -dimensional Laplacian operator in $\vec{\phi}$ space, we obtain a δ -function like topological tensor current

$$\tilde{j}^{\lambda\lambda_1 \dots \lambda_{\tilde{p}}} = \frac{1}{\sqrt{g}} \delta(\vec{\phi}) J^{\lambda\lambda_1 \dots \lambda_{\tilde{p}}} \left(\frac{\phi}{x} \right), \quad (47)$$

and find that $\tilde{j}^{\lambda\lambda_1 \dots \lambda_{\tilde{p}}} \neq 0$ only when $\vec{\phi} = 0$. So, it is essential to discuss the solutions of the equations

$$\phi^A(x) = 0, \quad A = 1, \dots, d. \quad (48)$$

This kind of solution plays an crucial role in realization of the \tilde{p} -brane scenario.

Suppose that the vector field $\vec{\phi}(x)$ possesses ℓ isolated zeroes, according to the deduction of Ref.[29] and the implicit function theorem[39,40], when the zeroes are regular points of $\vec{\phi}$ -mapping, i.e., the rank of the Jacobian matrix $[\partial_\mu \phi^A]$ is d , the solutions of $\vec{\phi}(x) = 0$ can be parameterized as

$$x^\mu = z_i^\mu(u^0, u^1, \dots, u^{\tilde{p}}), \quad i = 1, \dots, \ell, \quad \mu = 1, \dots, D, \quad (49)$$

where the subscript i represents the i -th solution and the parameters $u^I (I = 0, 1, 2, \dots, \tilde{p})$ span a $(\tilde{p} + 1)$ -dimensional submanifold which is called the i -th singular submanifold N_i in the total spacetime manifold X . These spatial \tilde{p} -dimension singular submanifolds N_i are just the world volumes of the topological \tilde{p} -branes in M -theory. The number of solutions ℓ takes the role of the brane number. This is the novel result of the present work. Therefore by making use of *GBC* theorem and ϕ -mapping topological current theory, we have established a novel field theory of creating topological \tilde{p} -branes. We must pointed out that based on the description of \tilde{p} -branes as topological defects in space-time[7,12], the vector field $\phi^A(x)$ ($A = 1, \dots, d$) can be looked upon as the order parameter fields of \tilde{p} -branes.

3 Inner topological structure of \tilde{p} -branes

From above discussions, we see that the kernel of ϕ -mapping plays an crucial role in creating of topological \tilde{p} -branes. Here we will focus on the zero points of order parameter field ϕ and will search for the inner topological structure of \tilde{p} -branes. It can be proved that there exists a d -dimensional submanifold M_i in X with local parametric equation

$$x^\mu = x^\mu(v^1, \dots, v^d), \quad \mu = 1, \dots, D, \quad (50)$$

which is transversal to every N_i at the point p_i with metric

$$g_{\mu\nu} B_I^\mu B_a^\nu|_{p_i} = 0, \quad I = 0, 1, \dots, p, \quad a = 1, \dots, d, \quad (51)$$

where

$$\frac{\partial x^\mu}{\partial u^I} = B_I^\mu, \quad \frac{\partial x^\nu}{\partial v^a} = B_a^\nu, \quad \mu, \nu = 1, 2, \dots, D, \quad (52)$$

are tangent vectors of N_i and M_i respectively. As we have pointed out in Ref.[19], the unit vector field defined in (38) gives a Gauss map $n : \partial M_i \rightarrow S^{d-1}$, and the generalized Winding Number can be given by the map

$$W_i = \frac{1}{A(S^{d-1})(d-1)!} \int_{\partial M_i} n^*(\varepsilon_{A_1 \dots A_d} n^{A_1} dn^{A_2} \wedge \dots \wedge dn^{A_d}). \quad (53)$$

where n^* denotes the pull back of map n and ∂M_i is the boundary of the neighborhood M_i of p_i on X with $p_i \notin \partial M_i$, $M_i \cap M_j = \emptyset$. It means that, when the point v^b covers ∂M_i once, the unit vector n will cover the region

∂M_i whose area is W_i times of $A^{(d-1)}$, i.e., the unit vector n will cover the unit sphere $A^{(d-1)}$ for W_i times. Using the Stokes' theorem in exterior differential form and duplicating the derivation of (47), we obtain

$$W_i = \int_{M_i} \delta(\vec{\phi}(v)) J\left(\frac{\phi}{v}\right) d^d v \quad (54)$$

where $J(\frac{\phi}{v})$ is the usual Jacobian determinant of $\vec{\phi}$ with respect to v

$$\varepsilon^{A_1 \cdots A_d} J\left(\frac{\phi}{v}\right) = \varepsilon^{\mu_1 \cdots \mu_d} \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \cdots \partial_{\mu_d} n^{A_d}. \quad (55)$$

According to the δ -function theory[41] and the ϕ -mapping theory, we know that $\delta(\vec{\phi})$ can be expanded as

$$\delta(\vec{\phi}) = \sum_{i=1}^{\ell} c_i \delta(N_i) \quad (56)$$

where the coefficients c_i must be positive, i.e., $c_i = |c_i|$. $\delta(N_i)$ is the δ -function in X on a submanifold N_i , [41,42]

$$\delta(N_i) = \int_{N_i} \delta^D(x - z_i(u)) \sqrt{g_u} d^{(\tilde{p}+1)} u, \quad i = 1, \cdots, \ell, \quad (57)$$

where $g_u = \det(g_{IJ})$. Substituting (56) into (54), and calculating the integral, we get the expression of c_i ,

$$c_i = \frac{\beta_i}{\left|J\left(\frac{\phi}{v}\right)\right|_{p_i}} = \frac{\beta_i \eta_i}{J\left(\frac{\phi}{v}\right)|_{p_i}}, \quad (58)$$

where the positive integer $\beta_i = |W_i|$ is called the Hopf index of ϕ -mapping on M_i , and $\eta_i = \text{sgn}(J(\frac{\phi}{v}))|_{p_i} = \pm 1$ is the Brouwer degree[19,43]. So we find the relations between the Hopf index β_i , the Brouwer degree η_i , and the winding number W_i

$$W_i = \beta_i \eta_i. \quad (59)$$

Therefore, the general topological current of the \tilde{p} -branes can be expressed directly as

$$\begin{aligned} \tilde{j}^{\lambda \lambda_1 \cdots \lambda_{\tilde{p}}} &= \frac{1}{\sqrt{g}} J^{\lambda \lambda_1 \cdots \lambda_{\tilde{p}}} \left(\frac{\phi}{x}\right) \\ &\cdot \sum_{i=1}^{\ell} \beta_i \eta_i \int_{N_i} \delta^D(x - z_i(u)) \sqrt{g_u} d^{(\tilde{p}+1)} u. \end{aligned} \quad (60)$$

From the above equation, we conclude that the $(\tilde{p} + 1)$ -dimensional singular submanifolds N_i ($i = 1, 2, \dots, \ell$) are world volumes of \tilde{p} -branes, and the inner topological structure of \tilde{p} -branes current is labelled by the total expansion of $\delta(\vec{\phi})$ which includes the topological information $\beta_i \eta_i$. In detail, β_i characterizes the absolute value of the topological charge of every \tilde{p} -brane and $\eta_i = +1$ corresponds to \tilde{p} -brane while $\eta_i = -1$ to anti \tilde{p} -brane. Taking the parametre u^0 and u^I ($I = 1, 2, \dots, \tilde{p}$) as the timelike evolution parameter and spacelike parameters respectively, the topological current of \tilde{p} -branes just represents \tilde{p} -dimensional topological defects with topological charges $\beta_i \eta_i$ moving in the D -dimensional total manifold X .

If we define a Lagrangian of \tilde{p} -brane as

$$L = \sqrt{\frac{1}{(\tilde{p} + 1)!} g_{\mu_0 \nu_0} g_{\mu_1 \nu_1} \cdots g_{\mu_{\tilde{p}} \nu_{\tilde{p}}} \tilde{j}^{\mu_0 \mu_1 \cdots \mu_{\tilde{p}}} \tilde{j}^{\nu_0 \nu_1 \cdots \nu_{\tilde{p}}}}, \quad (61)$$

which is just the generalization of Nielsen's Lagrangian of string[44] and includes the total information of arbitrary dimensional \tilde{p} -branes in X . It obvious that the Euler equations corresponding to the Lagrangian (61) will give the dynamics of the \tilde{p} -branes. From the above deductions, we can prove that

$$L = \frac{1}{\sqrt{g}} \delta(\vec{\phi}(x)). \quad (62)$$

Then, the action takes the form

$$\begin{aligned} S &= \int_X L \sqrt{g} d^D x = \int_X \delta(\vec{\phi}(x)) d^D x \\ &= \sum_{i=1}^l \beta_i \eta_i \int_{N_i} \sqrt{g_u} d^{(\tilde{p}+1)} u, \end{aligned} \quad (63)$$

i.e.

$$S = \sum_{i=1}^l \eta_i S_i, \quad (64)$$

where $S_i = \beta_i \int_{N_i} \sqrt{g_u} d^{(\tilde{p}+1)} u$. This is just the straightforward generalized Nambu action for the string world-sheet action [45]. Here this action for multi \tilde{p} -branes is obtained directly by ϕ -mapping theory, and it is easy to see that this action is just Nambu action for multistrings when $D - d = 2$. [29].

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